1 Cvičení

Příklad 1. $\mathfrak{so}(3,\mathbb{C}) \sim \mathfrak{sl}(2,\mathbb{C}) : [L_3, L_\pm] = \pm L_\pm,\ [L_+, L_-] = 2L_3,$

$$\rho(L_3) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \rho(L_+) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \rho(L_-) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$\rho(L_3)\,|\uparrow\rangle = \frac{1}{2}\,|\uparrow\rangle, \qquad \rho(L_3)\,|\downarrow\rangle = -\frac{1}{2}\,|\downarrow\rangle, \qquad \text{váhy: } \lambda = \pm\frac{1}{2},$$

$\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}\left(D^{1/2}\right),\ D^{1/2} = \operatorname{span}\{|\uparrow\rangle, |\downarrow\rangle\}$. Tenzorový součin ρ se sebou samou:

$$(\rho\otimes\rho)(L_3) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\otimes\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\otimes\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$(\rho\otimes\rho)(L_3)\,|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \qquad (\rho\otimes\rho)(L_3)\,|\uparrow\downarrow\rangle = \frac{1}{2}\,|\uparrow\downarrow\rangle - \frac{1}{2}\,|\uparrow\downarrow\rangle = 0$$
$$(\rho\otimes\rho)(L_3)\,|\downarrow\downarrow\rangle = -\,|\downarrow\downarrow\rangle \qquad (\rho\otimes\rho)(L_3)\,|\downarrow\uparrow\rangle = 0$$

$$(\rho\otimes\rho)(L_-)\,|\uparrow\uparrow\rangle = |\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle \qquad (\rho\otimes\rho)(L_-)\big(\,|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\big) = |\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle = 0$$
$$(\rho\otimes\rho)(L_-)\,|\downarrow\downarrow\rangle = 0$$

$$(\rho\otimes\rho)(L_+)\ldots$$

Váhy: $\pm 2\lambda, 0;\ n_{\pm 2\lambda} = 1,\ n_0 = 2.$

Příklad 2. $A_l = \mathfrak{sl}(l+1,\mathbb{C}) = \left\{A \in \mathbb{C}^{l+1,l+1}\middle|\, \mathrm{Tr}A = 0\right\}$

- Kořeny: $\mathfrak{g}_0 = \mathrm{diag} \subset \mathfrak{sl}(l+1),\ \dim\mathfrak{g}_0 = l, [\mathfrak{g}_0,\mathfrak{g}_0] = 0 \quad\Rightarrow\quad \mathfrak{g}_0$ Abelovská $\Rightarrow\quad \mathfrak{g}_0$ nilpotentní, tj. opravdu je to Cartanova podalgebra. Mějme

$$E_{ij} = \begin{matrix} \\ i\end{matrix}\begin{matrix}\begin{matrix} \\ \end{matrix} j\end{matrix}\begin{pmatrix} & \vdots\\ \ldots & 1\end{pmatrix}, \qquad i \neq j$$

  $\Rightarrow\quad \mathfrak{sl}(l+1) = \mathfrak{g}_0 + \operatorname{span}\{E_{ij}\}$ a pro $D \in \mathfrak{g}_0,\ D = \mathrm{diag}(d_1,\ldots,d_{l+1})$ máme $[D, E_{ij}] - (d_i - d_j)E_{ij}$. Nechť $\phi_j \in \mathfrak{sl}^*(l+1),\ \phi_j(D) = d_j \quad\Rightarrow\quad (\phi_i - \phi_j)(D)E_{ij} = [D, E_{ij}]$, tj:

$$\Delta = \left\{(\phi_i - \phi_j)\middle|\, i \neq j,\ i,j \in \widehat{l+1}\right\}$$

  Zvolíme $H_0 = \mathrm{diag}(h_1,\ldots,h_{l+1}),\ h_i > h_{i+1},\ (\phi_i - \phi_j)(H_0) \neq 0$, máme tedy uspořádání kořenů:

$$\phi_1 > \phi_2 > \cdots > \phi_{l+1} > 0.$$

$$\Delta^+ = \left\{\phi_i - \phi_j \middle| i < j \leqslant l+1\right\}$$
$$\Delta^p = \left\{\underbrace{\phi_i - \phi_{i+1}}_{=:\alpha_i}\middle| i \in \widehat{l}\right\}$$

  Ověříme, že pomocí Δ^p můžeme nakombinovat celé Δ:

$$\phi_i - \phi_j = (\phi_i - \phi_{i+1}) + (\phi_{i+1} - \phi_{i+2}) + \cdots + (\phi_{j-1} - \phi_j).$$

- Cartanova matice, Dynkinův diagram:

$$a_{\beta\alpha} = -(p+q) \overset{\alpha,\beta\in\Delta^p}{=} -q, \qquad\qquad \{\beta + k\alpha\}_{k=p}^{q} \in \Delta^+$$

$$\left.\begin{array}{l}\alpha_i := \phi_i - \phi_{i+1} \\ \alpha_j := \phi_j - \phi_{j+1}\end{array}\right\} \quad \Rightarrow \quad \alpha_i + k\alpha_j = \phi_i - \phi_{i+1} + k(\phi_j - \phi_{j+1}) \overset{!}{=} \phi_a - \phi_b,\ a < b$$

$$\begin{array}{llll}(i < j-1) \vee (i > j-1) & \Rightarrow & k = 0 & \Rightarrow \quad a_{ij} = 0 \\ (i = j-1) \vee (j = i-1) & \Rightarrow & k = 0 \vee k = 1 & \Rightarrow \quad a_{ij} = -1\end{array}$$

$$a = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 2 \end{pmatrix}, \qquad\qquad \underset{1}{\bullet} \text{—} \underset{2}{\bullet} \text{—} \cdots \text{—} \underset{l-1}{\bullet} \text{—} \underset{l}{\bullet}$$

- Adjungovaná reprezentace: váhy (kořeny): $\alpha_i = \phi_i - \phi_{i+1}$, $\alpha_i(T_j) = a_{ij}$, kde

$$\phi_i \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{l+1} \end{pmatrix} = d_i, \qquad\qquad \phi_1 > \phi_2 > \cdots > \phi_{l+1} > 0.$$

Z tvaru vah $\alpha_i = \phi_i - \phi_j$ a uspořádání ϕ_i plyne, že nejvyšší váha je $\phi_1 - \phi_{l+1} = \alpha_1 + \cdots + \alpha_l$.
K nalezení T_j využijeme $\alpha_i(T_j) = a_{ij} = t_{j,i} - t_{j,i+1} \neq 0$ pro $i = j-1, j, j+1$:

$$\left.\begin{array}{rl}\alpha_{j-1}(T_j) & = t_{j,j-1} - t_{j,j} = -1 \\ \alpha_j(T_j) & = t_{j,j} - t_{j,j+1} = 2 \\ \alpha_{j+1}(T_j) & = t_{j,j+1} - t_{j,j+2} = -1\end{array}\right\} \quad \Rightarrow \quad T_j = \begin{pmatrix} \ddots & & & & & & \\ & 0 & & & & & \\ & & 1 & \cdots & \cdots & \cdots & \\ & & & -1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \end{pmatrix} \quad j$$

- Fundamentální váhy, $\lambda_i(T_j) = \delta_{ij}$:

$$\lambda_1 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} = 1, \qquad \lambda_1 \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix} = 0 \qquad \Rightarrow \quad \lambda_1 = \phi_1$$

$$\lambda_2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{pmatrix} = 0, \qquad \lambda_2 \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \\ & & & & \ddots \end{pmatrix} = 1, \qquad \lambda_2 \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 \\ & & & & & \ddots \end{pmatrix} = 0$$

$\Rightarrow \quad \lambda_2 = \phi_2 + \phi_1 \quad \Rightarrow \quad \ldots \quad \Rightarrow \quad \lambda_i = \phi_1 + \cdots + \phi_i$. Je vidět že pak platí $\lambda_i(T_j) = \delta_{ij}$.

- Definující reprezentace: Mějme definující reprezentaci v standardní bázi (e_j), $D \in \mathfrak{g}_0$, $De_j = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{l+1} \end{pmatrix} e_j = d_j e_j$. Její váhy $\{\phi_1, \dots, \phi_{l+1}\}$, $\phi_{l+1} = -(\phi_1 + \dots + \phi_l)$, lze zapsat jako $\{\phi_1, \phi_1 - \alpha_1, \phi_1 - \alpha_1 - \alpha_2, \dots, \phi_1 - \alpha_1 - \dots - \alpha_l\}$. Nejvyšší váha je $\phi_1 = \lambda_1$, násobnosti 1, $\dim \rho_1 = l + 1$. $\rho_1 \wedge \rho_1$:

  $$(\rho_1 \wedge \rho_1)(e_i \wedge e_j) = (D \otimes \mathbb{1} + \mathbb{1} \otimes D)(e_i \otimes e_j - e_j \otimes e_i) = \\ = d_i e_i \otimes e_j - d_j e_j \otimes e_i + e_i \otimes d_j e_j - e_j \otimes d_i e_i = (d_i + d_j)(e_i \wedge e_j),$$

  váhy: $\{\phi_i + \phi_j | i \neq j\}$, $\dim \rho \wedge \rho = \binom{l+1}{2}$, nejvyšší je $\phi_1 + \phi_2$.

  Pro $\rho^{\wedge j}$ jsou váhy $\left\{\phi_{i_1} + \dots + \phi_{i_j} \middle| i_1 < \dots < i_j\right\}$, $\dim \rho^{\wedge j} = \binom{l+1}{j}$, nejvyšší váha $\lambda_j = \phi_1 + \dots + \phi_j$.

  Pro $\rho^{\wedge l}$ jsou váhy $\{\sum_{i \neq 1} \phi_i, \dots, \sum_{i \neq l+1} \phi_i\} = \{-\phi_1, \dots, -\phi_{l+1}\} \overset{l \neq 1}{\neq} \{\phi_1, \dots, \phi_{l+1}\}$. Takže nejvyšší váha je $-\lambda_{l+1}$. Když $l = 1$, pak $\rho^{\wedge l=1} \simeq \rho$, tj. $\rho^{\wedge l=1}$ je izomorfní definující reprezentaci.

Poznámka 1. Nechť ρ reprezentace $\mathfrak{g}$ na V, definujeme $\rho^T : \rho^T(X) = (-\rho(X))^T \quad \Rightarrow \quad \rho^{\wedge l} = \rho^T$.

Příklad 3. $C_l = \mathfrak{sp}(2l, \mathbb{C}) = \left\{A \in \mathbb{C}^{2l,2l} \middle| JA + A^T J = 0\right\}$, kde $J = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$

- Cartanova podalgebra: Označme $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

  $$JA + A^T J = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} + \begin{pmatrix} c^T & -a^T \\ d^T & -b^T \end{pmatrix} = 0 \qquad \Rightarrow \qquad d = -a^T,\ b = b^T,\ c = c^T$$

  $$\mathfrak{g}_0 = \left\{ \left(\begin{smallmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{smallmatrix}\right) \middle| \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_l) \in \mathbb{C}^{l,l} \right\}$$

  $$[\Lambda, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \qquad \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix}\right] = (\lambda_i - \lambda_j) \underbrace{\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix}}_{=: I_{ij},\ i \neq j}$$

  $$\left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \Lambda(E_{ij} + E_{ji}) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & (E_{ij} + E_{ji})\Lambda \\ 0 & 0 \end{pmatrix} = (\lambda_i + \lambda_j) \underbrace{\begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}}_{=: F_{ij},\ i \leqslant j}$$

  $$G_{ij} := F_{ij}^T \qquad \Rightarrow \qquad \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, G_{ij}\right] = -\left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, F_{ij}\right]^T = -(\lambda_i + \lambda_j) G_{ij}$$

  $\Rightarrow$ $\mathfrak{g}_0$ je skutečně Cartanova podalgebra. $\phi_i \left(\begin{smallmatrix} \Lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right) := \lambda_i$, $i \in \widehat{l}$ tvoří bázi $\mathfrak{g}_0^*$.

- Kořeny:

  $$\Delta = \{\phi_i - \phi_j | i \neq j\} \cup \{\phi_i + \phi_j | i \leqslant j\} \cup \{-(\phi_i + \phi_j) | i \leqslant j\}$$

  $H_0 :\ \phi_i(H_0) > \phi_{i+1}(H_0) > 0, \forall i.$

  $$\Delta^+ = \{\phi_i - \phi_j | i < j\} \cup \{\phi_i + \phi_j | i \leqslant j\}$$
  $$\Delta^- = \{\phi_i - \phi_j | i > j\} \cup \{-(\phi_i + \phi_j) | i \leqslant j\}$$
  $$\Delta^p = \{ \underbrace{\phi_i - \phi_{i+1}}_{=: \alpha_i} | i \in \widehat{l-1} \} \cup \{ \underbrace{2\phi_l}_{=: \alpha_l} \}$$

$$\phi_i - \phi_j = (\phi_i - \phi_{i+1}) + \cdots + (\phi_{j-1} - \phi_j) = \sum_{k=1}^{j-1} \alpha_k$$

$$\phi_i + \phi_j = 2\phi_l + (\phi_i - \phi_l) + (\phi_j - \phi_l) = 2\phi_l + \sum_{k=i}^{l-1} \alpha_k + \sum_{k=j}^{l-1} \alpha_k$$

$a_{\beta\alpha} \overset{\alpha,\beta\in\Delta^p}{=} -q$:

$$\begin{array}{lllll}
\{\alpha_i + k\alpha_j\}_{i,j<l} & = (\phi_i - \phi_{i+1}) + k(\phi_j - \phi_{j+1}) & \Rightarrow & |i-j| > 1 & \Rightarrow \quad k = 0 \\
 & & & |i-j| = 1 & \Rightarrow \quad k = 0 \vee k = 1 \\
\{\alpha_i + k\alpha_l\}_{i<l} & = (\phi_i - \phi_{i+1}) + 2k\phi_l & \Rightarrow & i < l-1 & \Rightarrow \quad k = 0 \\
 & & & i = l-1 & \Rightarrow \quad k = 0 \vee k = 1 \\
\{\alpha_l + k\alpha_i\}_{i,j<l} & = 2\phi_l + k(\phi_i - \phi_{i+1}) & \Rightarrow & i < l-1 & \Rightarrow \quad k = 0 \\
 & & & i = l-1 & \Rightarrow \quad k = 0 \vee k = 1 \vee k = 2
\end{array}$$

$$\Rightarrow \quad a_{l-1,l} = -1 = \frac{\langle \alpha_{l-1}, \alpha_l \rangle}{\langle \alpha_l, \alpha_l \rangle},\ a_{l,l-1} = -2 = \frac{\langle \alpha_l, \alpha_{l-1} \rangle}{\langle \alpha_{l-1}, \alpha_{l-1} \rangle} \quad \Rightarrow \quad \|\alpha_l\| = \sqrt{2}\,\|\alpha_{l-1}\|.$$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}, \qquad \overset{1}{\bullet} \!\!-\!\!\!- \overset{2}{\bullet} \!\!-\!\!\!- \cdots \!\!-\!\!\!- \overset{l-2}{\bullet} \!\!-\!\!\!- \overset{l-1}{\bullet} \!\!\Longrightarrow\!\! \overset{l}{\bullet}$$

- Definující reprezentace: $D \in \mathfrak{g}_0,\ \phi_i(D) = d_i$:

$$D = \begin{pmatrix} d_1 & & & & & & \\ & \ddots & & & & & \\ & & d_l & & & & \\ & & & -d_1 & & & \\ & & & & \ddots & & \\ & & & & & -d_l \end{pmatrix}$$

  Definující reprezentace má váhy $\{\phi_1, \ldots, \phi_l, \phi_{-1}, \ldots, \phi_{-l}\},\ \dim = 2l$, nejvyšší váha je ϕ_1.

- Adjungovaná reprezentace: $\alpha_i = \phi_i - \phi_{i+1},\ i \leqslant l-1,\ \alpha_l = 2\phi_l,\ \alpha_i(T_j) = a_{ij}$

$$T_j = \begin{pmatrix} \ddots & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -1 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 1 & \cdots & \cdots & \cdots \\ & & & & & & & & -1 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & \ddots \end{pmatrix} \begin{matrix} \\ \\ j \\ \\ \\ \\ \\ l+j \\ \\ \\ \end{matrix} \qquad j \leqslant l-1$$

$$\left.\begin{array}{rl} \alpha_i(T_l) & = 0,\ i < l-1 \\ \alpha_{l-1}(T_l) & = -1 \\ \alpha_l(T_l) & = 2 \end{array}\right\} \quad \Rightarrow \quad T_l = \begin{pmatrix} \ddots & & & & & & \\ & 0 & & & & & \\ & & 1 & \dots & \dots & \dots & \dots \\ & & & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \\ & & & & & & 1 \end{pmatrix} \quad l$$

$\lambda_i(T_j) = \delta_{ij} \quad \Rightarrow \quad \lambda_i = \phi_1 + \dots + \phi_i,\ i \in \hat{I}.$

Příklad 4. $D_l = \mathfrak{so}(2l, \mathbb{C}) = \left\{ A \in \mathbb{C}^{2l,2l} \middle| A^T J + JA = 0 \right\}$, kde $J = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$, $l > 1$

Označme $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$A^T J + JA = \begin{pmatrix} c^T & a^T \\ d^T & b^T \end{pmatrix} + \begin{pmatrix} c & d \\ a & b \end{pmatrix} = 0 \qquad \Rightarrow \qquad d = -a^T,\ b = -b^T,\ c = -c^T$$

- Cartanova podalgebra: Ukážeme že $\mathfrak{g}_0 = \{H = \mathrm{diag}(\lambda_1\sigma_2, \dots, \lambda_l\sigma_2)\}$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Nechť $X \in \mathbb{C}^{2,2}$, $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$

  $$\lambda_i\sigma_2 X - \lambda_j X\sigma_2 = i\lambda_i \begin{pmatrix} -x_{21} & -x_{22} \\ x_{11} & x_{12} \end{pmatrix} - i\lambda_j \begin{pmatrix} x_{12} & -x_{11} \\ x_{22} & -x_{21} \end{pmatrix} = c(\lambda_i, \lambda_j) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

  Zapíšeme ve tvaru:

  $$i \begin{pmatrix} ic & -\lambda_j & -\lambda_i & 0 \\ \lambda_j & ic & 0 & -\lambda_i \\ \lambda_i & 0 & ic & -\lambda_j \\ 0 & \lambda_i & \lambda_j & ic \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = 0$$

  Z požadavku řešitelnosti soustavy ($\det = 0$) dostaneme $c_{1,2,3,4} = \pm(\lambda_i \pm \lambda_j)$. Pro $c_1 = \lambda_i + \lambda_j$ najdeme $X_1 = \left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) = \sigma_3 + i\sigma_1$.

  $$\widetilde{F} := X_1, \qquad F_{ij} := \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} & \\ \vdots & \vdots \\ \cdots & \widetilde{F} \\ \cdots & -\widetilde{F}^T \\ & \end{pmatrix} \end{array}, \quad i < j, \qquad [H, F_{ij}] = (\lambda_i + \lambda_j) F_{ij} \overset{\exists i,j}{\neq} 0$$

  $$\left[H, F_{ij}^+\right] = \left[H^+, F_{ij}^+\right] = -\left[H, F_{ij}\right]^+ = -(\lambda_i + \lambda_j) F_{ij}^+$$

  Pro $c_2 = \lambda_i - \lambda_j$ dostaneme:

  $$\widetilde{G} := \mathbb{1} + \sigma_2, \qquad G_{ij} := \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} & \\ \vdots & \vdots \\ \cdots & \widetilde{G} \\ \cdots & -\widetilde{G}^T \\ & \end{pmatrix} \end{array}, \quad i < j$$

  $$[H, G_{ij}] = (\lambda_i + \lambda_j) G_{ij}, \qquad [H, G_{ij}] = (\lambda_i + \lambda_j) G_{ij}$$

- Kořeny: $\phi_j \in g_0^*$, $\phi_j(H) = \lambda_j$:

$$\Delta = \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i \neq j\} \cup \{-(\phi_i + \phi_j) | i < j\}$$

$H_0 = \operatorname{diag}(\lambda_1, \ldots, \lambda_l)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$:

$$\Delta^+ = \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i < j\}$$
$$\Delta^p = \{\underbrace{\phi_i - \phi_{i+}}_{=:\alpha_i} | i \in \widehat{l-i}\} \cup \{\underbrace{\phi_{l-1} + \phi_l}_{=:\alpha_l}\}$$

$$\begin{array}{lll} \alpha_i + k\alpha_{i+1} = (\phi_i - \phi_{i+1}) + k(\phi + 1 - \phi_{i+2}),\ i \in \widehat{l-1} & \Rightarrow & k = 0 \vee k = 1 \\ \alpha_{l-2} + k\alpha_l = (\phi_{l-2} - \phi_{l-1}) + k(\phi_{l-1} + \phi_l) & \Rightarrow & k = 0 \vee k = 1 \\ \alpha_{l-1} + k\alpha_l = (\phi_{l-1} - \phi_l) + k(\phi_{l-1} + \phi_l) & \Rightarrow & k = 0 \end{array}$$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix}, \qquad \begin{array}{ccccccccc} \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \\ 1 & & 2 & & & & l-3 & & l-2 \end{array} \begin{array}{l} \nearrow\, l-1 \\ \searrow\, l \end{array}$$

- Váhy:

$$H = \begin{pmatrix} d_1\sigma_2 & & \\ & \ddots & \\ & & d_l\sigma_2 \end{pmatrix} = H(d_1, \ldots, d_l), \qquad \begin{array}{l} \phi_i(H) = d_i \\ \alpha_i = \phi_i - \phi_{i+1},\ i \leqslant l-1 \\ \alpha_l = \phi_{l-1} + \phi_l \\ T_i = H(0, \ldots, 0, \underset{i}{1}, \underset{i+1}{-1}, 0, \ldots, 0),\ i \leqslant l-1x \end{array}$$

T_l:

$$\left.\begin{array}{rll} \alpha_{l-2}(T_l) & = -1 & = d_{l-2} - d_{l-1} \\ \alpha_{l-1}(T_l) & = 0 & = d_{l-1} - d_l \\ \alpha_l(T_l) & = 2 & = \phi_{l-1}(T_l) + \phi_l(t_l) = d_{l-1} + d_l \end{array}\right\} \quad \Rightarrow \quad T_l = H(0, \ldots, 0, 1, 1)$$

$\lambda_i(T_j) = \delta_{ij}$:

$$\begin{aligned} \lambda_1 &= \phi_1 \\ \lambda_i &= \phi_1 + \cdots + \phi_i,\ i \leqslant l-2 \\ \lambda_{l-1} &= \frac{1}{2}(\phi_1 + \cdots + \phi_{l-1} - \phi_l) \\ \lambda_l &= \frac{1}{2}(\phi_1 + \cdots + \phi_l) \end{aligned}$$

Definující reprezentace má váhy $\{\phi_1, \ldots, \phi_l, -\phi_1, \ldots, -\phi_l\}$.

Příklad 5. $B_l = \mathfrak{so}(2l+1, \mathbb{C})$

$$\mathfrak{g}_0 = \left\{ H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} \right\}, \quad \phi_i H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} = \lambda_i, \quad X := \left(\begin{array}{ccc|c} & & & v \\ & & & \\ \hline v^T & & & 0 \end{array}\right)$$

$$[H,X] = \left(\begin{array}{c|c} & \lambda_i\sigma_1 v \\ \hline & 0 \end{array}\right) - \left(\begin{array}{c|c} & v \\ \hline -\lambda_i(\sigma_1 v)^T & 0 \end{array}\right) = \lambda_i \left(\begin{array}{c|c} & \sigma_1 v \\ \hline (\sigma_1 v)^T & 0 \end{array}\right)$$

Za v můžeme volit vlastní vektory σ_1. Dále zvolíme $H_0:\ \lambda_1 > \cdots > \lambda_l,\ \lambda_i = \phi(H_0)$.

$$\begin{aligned}
\Delta &= \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i \neq j\} \cup \{-(\phi_i + \phi_j) | i < j\} \cup \{\phi_i\} \cup \{-\phi_i\} \\
\Delta^+ &= \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i < j\} \cup \{\phi_i\} \\
\Delta^p &= \{\underbrace{\phi_i - \phi_{i+1}}_{=:\alpha_i} | i \in \widehat{l-1}\} \cup \{\underbrace{\phi_l}_{=:\alpha_l}\}
\end{aligned}$$

$$\begin{array}{lll}
\alpha_{l-2} + k\alpha_l &= (\phi_{l-2} - \phi_{l-1}) + k\phi_l & \Rightarrow \quad k = 0 \\
\alpha_{l-1} + k\alpha_l &= (\phi_{l-1} - \phi_l) + k\phi_l & \Rightarrow \quad k = 0 \vee k = 1 \vee k = 2 \\
\alpha_l + k\alpha_{l-1} &= \phi_l + k(\phi_{l-1} - \phi_l) & \Rightarrow \quad k = 0 \vee k = 1
\end{array}$$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}, \qquad \begin{array}{ccccccccc} \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \Longleftarrow \bullet \\ 1 & & 2 & & & & l-2 & & l-1 \quad l \end{array}$$

$$H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} \qquad \begin{array}{l} \phi_i(H) = d_i \\ \alpha_i = \phi_i - \phi_{i+1},\ i \leqslant l-1 \\ \alpha_l = \phi_l \\ T_i = H(0,\ldots,0,\underset{i}{1},\underset{i+1}{-1},0,\ldots,0) \end{array}$$

T_l:

$$\left.\begin{array}{rl} \alpha_{l-1}(T_l) &= -2 \\ \alpha_l(t_l) &= 2 \end{array}\right\} \quad \Rightarrow \quad T_l = H(0,\ldots,0,2)$$

$\lambda_i(T_j) = \delta_{ij}$:

$$\begin{aligned}
\lambda_i &= \phi_1 + \cdots + \phi_i,\ i \leqslant l-1 \\
\lambda_l &= \frac{1}{2}(\phi_1 + \cdots + \phi_l)
\end{aligned}$$