1 Cvičení

Příklad 1. $\mathfrak{so}(3,\mathbb{C}) \sim \mathfrak{sl}(2,\mathbb{C}) : [L_3, L_\pm] = \pm L_\pm,\ [L_+, L_-] = 2L_3,$

$$\rho(L_3) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \rho(L_+) = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \qquad \rho(L_-) = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$
$$\rho(L_3)\left|\uparrow\right\rangle = \frac{1}{2}\left|\uparrow\right\rangle, \qquad \rho(L_3)\left|\downarrow\right\rangle = -\frac{1}{2}\left|\downarrow\right\rangle, \qquad \text{váhy: } \lambda = \pm\frac{1}{2},$$

$\rho : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}\left(D^{1/2}\right),\ D^{1/2} = \operatorname{span}\{\left|\uparrow\right\rangle, \left|\downarrow\right\rangle\}$. Tenzorový součin ρ se sebou samou:

$$(\rho\otimes\rho)(L_3) = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$(\rho\otimes\rho)(L_3)\left|\uparrow\uparrow\right\rangle = \left|\uparrow\uparrow\right\rangle \qquad (\rho\otimes\rho)(L_3)\left|\uparrow\downarrow\right\rangle = \frac{1}{2}\left|\uparrow\downarrow\right\rangle - \frac{1}{2}\left|\uparrow\downarrow\right\rangle = 0$$
$$(\rho\otimes\rho)(L_3)\left|\downarrow\downarrow\right\rangle = -\left|\downarrow\downarrow\right\rangle \qquad (\rho\otimes\rho)(L_3)\left|\downarrow\uparrow\right\rangle = 0$$

$$(\rho\otimes\rho)(L_-)\left|\uparrow\uparrow\right\rangle = \left|\downarrow\uparrow\right\rangle + \left|\uparrow\downarrow\right\rangle \qquad (\rho\otimes\rho)(L_-)\big(\left|\downarrow\uparrow\right\rangle - \left|\uparrow\downarrow\right\rangle\big) = \left|\downarrow\downarrow\right\rangle - \left|\downarrow\downarrow\right\rangle = 0$$
$$(\rho\otimes\rho)(L_-)\left|\downarrow\downarrow\right\rangle = 0$$

$$(\rho\otimes\rho)(L_+)\ldots$$

Váhy: $\pm 2\lambda, 0$; $n_{\pm 2\lambda} = 1,\ n_0 = 2$.

Příklad 2. $A_l = \mathfrak{sl}(l+1,\mathbb{C}) = \left\{A \in \mathbb{C}^{l+1,l+1} \middle| \operatorname{Tr} A = 0\right\}$

- Kořeny: $\mathfrak{g}_0 = \operatorname{diag} \subset \mathfrak{sl}(l+1)$, $\dim\mathfrak{g}_0 = l, [\mathfrak{g}_0, \mathfrak{g}_0] = 0 \quad\Rightarrow\quad \mathfrak{g}_0$ Abelovská $\Rightarrow\quad \mathfrak{g}_0$ nilpotentní, tj. opravdu je to Cartanova podalgebra. Mějme

$$E_{ij} = \begin{array}{c c} & j \\ & \\ i & \begin{pmatrix} & \vdots \\ \ldots & 1 \end{pmatrix} \end{array}, \qquad i \neq j$$

$\Rightarrow\quad \mathfrak{sl}(l+1) = \mathfrak{g}_0 + \operatorname{span}\{E_{ij}\}$ a pro $D \in \mathfrak{g}_0,\ D = \operatorname{diag}(d_1, \ldots, d_{l+1})$ máme $[D, E_{ij}] - (d_i - d_j)E_{ij}$. Nechť $\phi_j \in \mathfrak{sl}^*(l+1),\ \phi_j(D) = d_j \quad\Rightarrow\quad (\phi_i - \phi_j)(D)E_{ij} = [D, E_{ij}]$, tj:

$$\Delta = \left\{(\phi_i - \phi_j) \middle|\ i \neq j,\ i, j \in \widehat{l+1}\right\}$$

Zvolíme $H_0 = \operatorname{diag}(h_1, \ldots, h_{l+1}),\ h_i > h_{i+1},\ (\phi_i - \phi_j)(H_0) \neq 0$, máme tedy uspořádání kořenů:

$$\phi_1 > \phi_2 > \cdots > \phi_{l+1} > 0.$$

$$\Delta^+ = \{\phi_i - \phi_j | i < j \leqslant l+1\}$$
$$\Delta^p = \{\underbrace{\phi_i - \phi_{i+1}}_{=:\alpha_i} | i \in \hat{l}\}$$

Ověříme, že pomocí Δ^p můžeme nakombinovat celé Δ:

$$\phi_i - \phi_j = (\phi_i - \phi_{i+1}) + (\phi_{i+1} - \phi_{i+2}) + \cdots + (\phi_{j-1} - \phi_j).$$

- Cartanova matice, Dynkinův diagram:

$$a_{\beta\alpha} = -(p+q) \overset{\alpha,\beta\in\Delta^p}{=} -q, \qquad\qquad \{\beta + k\alpha\}_{k=p}^{q} \in \Delta^+$$

$$\left.\begin{aligned} \alpha_i &:= \phi_i - \phi_{i+1} \\ \alpha_j &:= \phi_j - \phi_{j+1} \end{aligned}\right\} \quad \Rightarrow \quad \alpha_i + k\alpha_j = \phi_i - \phi_{i+1} + k(\phi_j - \phi_{j+1}) \overset{!}{=} \phi_a - \phi_b,\ a < b$$

$$\begin{array}{lclcl} (i < j-1) \vee (i > j-1) & \Rightarrow & k = 0 & \Rightarrow & a_{ij} = 0 \\ (i = j-1) \vee (j = i-1) & \Rightarrow & k = 0 \vee k = 1 & \Rightarrow & a_{ij} = -1 \end{array}$$

$$a = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ & & -1 & 2 \end{pmatrix}, \qquad\qquad \begin{array}{ccccccccc} \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \\ 1 & & 2 & & & & l-1 & & l \end{array}$$

- Adjungovaná reprezentace: váhy (kořeny): $\alpha_i = \phi_i - \phi_{i+1}$, $\alpha_i(T_j) = a_{ij}$, kde

$$\phi_i \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{l+1} \end{pmatrix} = d_i, \qquad\qquad \phi_1 > \phi_2 > \cdots > \phi_{l+1} > 0.$$

  Z tvaru vah $\alpha_i = \phi_i - \phi_j$ a uspořádání ϕ_i plyne, že nejvyšší váha je $\phi_1 - \phi_{l+1} = \alpha_1 + \cdots + \alpha_l$.
  K nalezení T_j využijeme $\alpha_i(T_j) = a_{ij} = t_{j,i} - t_{j,i+1} \neq 0$ pro $i = j-1, j, j+1$:

$$\left.\begin{aligned} \alpha_{j-1}(T_j) &= t_{j,j-1} - t_{j,j} = -1 \\ \alpha_j(T_j) &= t_{j,j} - t_{j,j+1} = 2 \\ \alpha_{j+1}(T_j) &= t_{j,j+1} - t_{j,j+2} = -1 \end{aligned}\right\} \quad \Rightarrow \quad T_j = \begin{array}{cc} \begin{pmatrix} \ddots & & & & & \\ & 0 & & & & \\ & & 1 & \dots & \dots & \dots \\ & & & -1 & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} & \begin{array}{c} \\ \\ j \\ \\ \\ \\ \end{array} \end{array}$$

- Fundamentální váhy, $\lambda_i(T_j) = \delta_{ij}$:

$$\lambda_1 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = 1, \qquad \lambda_1 \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} = 0 \qquad\qquad \Rightarrow \quad \lambda_1 = \phi_1$$

$$\lambda_2 \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix} = 0, \qquad \lambda_2 \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} = 1, \qquad\qquad \lambda_2 \begin{pmatrix} \ddots & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} = 0$$

$\Rightarrow \quad \lambda_2 = \phi_2 + \phi_1 \quad \Rightarrow \quad \dots \quad \Rightarrow \quad \lambda_i = \phi_1 + \cdots + \phi_i$. Je vidět že pak platí $\lambda_i(T_j) = \delta_{ij}$.

- Definující reprezentace: Mějme definující reprezentaci v standardní bázi (e_j), $D \in \mathfrak{g}_0$, $De_j = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_{l+1} \end{pmatrix} e_j = d_j e_j$. Její váhy $\{\phi_1, \dots, \phi_{l+1}\}$, $\phi_{l+1} = -(\phi_1 + \dots + \phi_l)$, lze zapsat jako $\{\phi_1, \phi_1 - \alpha_1, \phi_1 - \alpha_1 - \alpha_2, \dots, \phi_1 - \alpha_1 - \dots - \alpha_l\}$. Nejvyšší váha je $\phi_1 = \lambda_1$, násobnosti 1, $\dim \rho_1 = l + 1$. $\rho_1 \wedge \rho_1$:

$$\begin{aligned}(\rho_1 \wedge \rho_1)(e_i \wedge e_j) &= (D \otimes \mathbb{1} + \mathbb{1} \otimes D)(e_i \otimes e_j - e_j \otimes e_i) = \\ &= d_i e_i \otimes e_j - d_j e_j \otimes e_i + e_i \otimes d_j e_j - e_j \otimes d_i e_i = (d_i + d_j)(e_i \wedge e_j),\end{aligned}$$

  váhy: $\{\phi_i + \phi_j | i \neq j\}$, $\dim \rho \wedge \rho = \binom{l+1}{2}$, nejvyšší je $\phi_1 + \phi_2$.

  Pro $\rho^{\wedge j}$ jsou váhy $\left\{\phi_{i_1} + \dots + \phi_{i_j} \middle| i_1 < \dots < i_j\right\}$, $\dim \rho^{\wedge j} = \binom{l+1}{j}$, nejvyšší váha $\lambda_j = \phi_1 + \dots + \phi_j$.

  Pro $\rho^{\wedge l}$ jsou váhy $\left\{\sum_{i \neq 1} \phi_i, \dots, \sum_{i \neq l+1} \phi_i\right\} = \{-\phi_1, \dots, -\phi_{l+1}\} \overset{l\neq 1}{\neq} \{\phi_1, \dots, \phi_{l+1}\}$. Takže nejvyšší váha je $-\lambda_{l+1}$. Když $l = 1$, pak $\rho^{\wedge l=1} \simeq \rho$, tj. $\rho^{\wedge l=1}$ je izomorfní definující reprezentaci.

Poznámka 1. Nechť ρ reprezentace $\mathfrak{g}$ na V, definujeme $\rho^T : \rho^T(X) = (-\rho(X))^T \quad \Rightarrow \quad \rho^{\wedge l} = \rho^T$.

Příklad 3. $C_l = \mathfrak{sp}(2l, \mathbb{C}) = \left\{A \in \mathbb{C}^{2l,2l} \middle| JA + A^T J = 0\right\}$, kde $J = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$

- Cartanova podalgebra: Označme $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$:

$$JA + A^T J = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} + \begin{pmatrix} c^T & -a^T \\ d^T & -b^T \end{pmatrix} = 0 \qquad \Rightarrow \qquad d = -a^T,\ b = b^T,\ c = c^T$$

$$\mathfrak{g}_0 = \left\{ \left(\begin{smallmatrix} \Lambda & 0 \\ 0 & -\Lambda \end{smallmatrix}\right) \middle| \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_l) \in \mathbb{C}^{l,l} \right\}$$

$$[\Lambda, E_{ij}] = (\lambda_i - \lambda_j) E_{ij} \qquad \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix} \right] = (\lambda_i - \lambda_j) \underbrace{\begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ij} \end{pmatrix}}_{=:I_{ij},\ i \neq j}$$

$$\left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & \Lambda(E_{ij} + E_{ji}) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & (E_{ij} + E_{ji})\Lambda \\ 0 & 0 \end{pmatrix} = (\lambda_i + \lambda_j) \underbrace{\begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}}_{=:F_{ij},\ i \leqslant j}$$

$$G_{ij} := F_{ij}^T \qquad \Rightarrow \qquad \left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, G_{ij} \right] = -\left[\begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, F_{ij} \right]^T = -(\lambda_i + \lambda_j) G_{ij}$$

  $\Rightarrow$ $\mathfrak{g}_0$ je skutečně Cartanova podalgebra. $\phi_i \left(\begin{smallmatrix} \Lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right) := \lambda_i$, $i \in \widehat{l}$ tvoří bázi $\mathfrak{g}_0^*$.

- Kořeny:

$$\Delta = \left\{\phi_i - \phi_j \middle| i \neq j\right\} \cup \left\{\phi_i + \phi_j \middle| i \leqslant j\right\} \cup \left\{-(\phi_i + \phi_j) \middle| i \leqslant j\right\}$$

  $H_0 :\ \phi_i(H_0) > \phi_{i+1}(H_0) > 0, \forall i$.

$$\begin{aligned} \Delta^+ &= \left\{\phi_i - \phi_j \middle| i < j\right\} \cup \left\{\phi_i + \phi_j \middle| i \leqslant j\right\} \\ \Delta^- &= \left\{\phi_i - \phi_j \middle| i > j\right\} \cup \left\{-(\phi_i + \phi_j) \middle| i \leqslant j\right\} \\ \Delta^p &= \{ \underbrace{\phi_i - \phi_{i+1}}_{=:\alpha_i} \big| i \in \widehat{l-1} \} \cup \{ \underbrace{2\phi_l}_{=:\alpha_l} \} \end{aligned}$$

$$\phi_i - \phi_j = (\phi_i - \phi_{i+1}) + \cdots + (\phi_{j-1} - \phi_j) = \sum_{k=1}^{j-1} \alpha_k$$

$$\phi_i + \phi_j = 2\phi_l + (\phi_i - \phi_l) + (\phi_j - \phi_l) = 2\phi_l + \sum_{k=i}^{l-1} \alpha_k + \sum_{k=j}^{l-1} \alpha_k$$

$a_{\beta\alpha} \overset{\alpha,\beta\in\Delta^p}{=} -q$:

$$\begin{array}{llllll}
\{\alpha_i + k\alpha_j\}_{i,j<l} & = (\phi_i - \phi_{i+1}) + k(\phi_j - \phi_{j+1}) & \Rightarrow & |i-j| > 1 & \Rightarrow & k = 0 \\
 & & & |i-j| = 1 & \Rightarrow & k = 0 \vee k = 1 \\
\{\alpha_i + k\alpha_l\}_{i<l} & = (\phi_i - \phi_{i+1}) + 2k\phi_l & \Rightarrow & i < l-1 & \Rightarrow & k = 0 \\
 & & & i = l-1 & \Rightarrow & k = 0 \vee k = 1 \\
\{\alpha_l + k\alpha_i\}_{i,j<l} & = 2\phi_l + k(\phi_i - \phi_{i+1}) & \Rightarrow & i < l-1 & \Rightarrow & k = 0 \\
 & & & i = l-1 & \Rightarrow & k = 0 \vee k = 1 \vee k = 2
\end{array}$$

$$\Rightarrow \quad a_{l-1,l} = -1 = \tfrac{\langle \alpha_{l-1},\alpha_l \rangle}{\langle \alpha_l,\alpha_l \rangle},\ a_{l,l-1} = -2 = \tfrac{\langle \alpha_l,\alpha_{l-1} \rangle}{\langle \alpha_{l-1},\alpha_{l-1} \rangle} \quad \Rightarrow \quad \|\alpha_l\| = \sqrt{2}\,\|\alpha_{l-1}\|.$$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}, \qquad \begin{array}{ccccccccccc} \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet & \Rightarrow & \bullet \\ 1 & & 2 & & & & l-2 & & l-1 & & l \end{array}$$

- Definující reprezentace: $D \in \mathfrak{g}_0$, $\phi_i(D) = d_i$:

$$D = \begin{pmatrix} d_1 & & & & & \\ & \ddots & & & & \\ & & d_l & & & \\ & & & -d_1 & & \\ & & & & \ddots & \\ & & & & & -d_l \end{pmatrix}$$

  Definující reprezentace má váhy $\{\phi_1, \ldots, \phi_l, \phi_{-1}, \ldots, \phi_{-l}\}$, $\dim = 2l$, nejvyšší váha je ϕ_1.

- Adjungovaná reprezentace: $\alpha_i = \phi_i - \phi_{i+1}$, $i \leqslant l-1$, $\alpha_l = 2\phi_l$, $\alpha_i(T_j) = a_{ij}$

$$T_j = \begin{pmatrix} \ddots & & & & & & & & & & \\ & 0 & & & & & & & & & \\ & & 1 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & -1 & & & & & & & \\ & & & & 0 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 0 & & & & \\ & & & & & & & 1 & \dots & \dots & \dots \\ & & & & & & & & -1 & & \\ & & & & & & & & & 0 & \\ & & & & & & & & & & \ddots \end{pmatrix} \begin{matrix} \\ \\ j \\ \\ \\ \\ \\ l+j \\ \\ \\ \\ \end{matrix} \qquad j \leqslant l-1$$

$$\left.\begin{aligned} \alpha_i(T_l) &= 0,\ i < l-1 \\ \alpha_{l-1}(T_l) &= -1 \\ \alpha_l(T_l) &= 2 \end{aligned}\right\} \quad \Rightarrow \quad T_l = \begin{pmatrix} \ddots & & & & & & & \\ & 0 & & & & & & \\ & & 1 & \dots & \dots & \dots & \dots & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & & \\ & & & & & & 1 & \end{pmatrix} \quad l$$

$$\lambda_i(T_j) = \delta_{ij} \quad \Rightarrow \quad \lambda_i = \phi_1 + \dots + \phi_i,\ i \in \hat{l}.$$

Příklad 4. $D_l = \mathfrak{so}(2l, \mathbb{C}) = \left\{ A \in \mathbb{C}^{2l,2l} \middle| A^T J + JA = 0 \right\}$, kde $J = \left(\begin{smallmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$, $l > 1$

Označme $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$A^T J + JA = \begin{pmatrix} c^T & a^T \\ d^T & b^T \end{pmatrix} + \begin{pmatrix} c & d \\ a & b \end{pmatrix} = 0 \qquad \Rightarrow \qquad d = -a^T,\ b = -b^T,\ c = -c^T$$

- Cartanova podalgebra: Ukážeme že $\mathfrak{g}_0 = \{ H = \mathrm{diag}(\lambda_1\sigma_2, \dots, \lambda_l\sigma_2) \}$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$. Nechť $X \in \mathbb{C}^{2,2}$, $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$

$$\lambda_i \sigma_2 X - \lambda_j X \sigma_2 = i\lambda_i \begin{pmatrix} -x_{21} & -x_{22} \\ x_{11} & x_{12} \end{pmatrix} - i\lambda_j \begin{pmatrix} x_{12} & -x_{11} \\ x_{22} & -x_{21} \end{pmatrix} = c(\lambda_i, \lambda_j) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

Zapíšeme ve tvaru:

$$i \begin{pmatrix} ic & -\lambda_j & -\lambda_i & 0 \\ \lambda_j & ic & 0 & -\lambda_i \\ \lambda_i & 0 & ic & -\lambda_j \\ 0 & \lambda_i & \lambda_j & ic \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \\ x_{21} \\ x_{22} \end{pmatrix} = 0$$

Z požadavku řešitelnosti soustavy (det $= 0$) dostaneme $c_{1,2,3,4} = \pm(\lambda_i \pm \lambda_j)$. Pro $c_1 = \lambda_i + \lambda_j$ najdeme $X_1 = \left(\begin{smallmatrix} 1 & i \\ i & -1 \end{smallmatrix}\right) = \sigma_3 + i\sigma_1$.

$$\widetilde{F} := X_1, \qquad F_{ij} := \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{ccc} & i & j \end{array} \\ \begin{pmatrix} & \vdots & \vdots \\ \dots & & \widetilde{F} \\ \dots & -\widetilde{F}^T & \\ & & \end{pmatrix} \end{array}, \quad i < j, \qquad [H, F_{ij}] = (\lambda_i + \lambda_j) F_{ij} \overset{\exists i,j}{\neq} 0$$

$$\left[H, F_{ij}^+ \right] = \left[H^+, F_{ij}^+ \right] = -\left[H, F_{ij} \right]^+ = -(\lambda_i + \lambda_j) F_{ij}^+$$

Pro $c_2 = \lambda_i - \lambda_j$ dostaneme:

$$\widetilde{G} := \mathbb{1} + \sigma_2, \qquad G_{ij} := \begin{array}{c} \\ \\ i \\ j \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{ccc} & i & j \end{array} \\ \begin{pmatrix} & \vdots & \vdots \\ \dots & & \widetilde{G} \\ \dots & -\widetilde{G}^T & \\ & & \end{pmatrix} \end{array}, \quad i < j$$

$$[H, G_{ij}] = (\lambda_i + \lambda_j) G_{ij}, \qquad [H, G_{ij}] = (\lambda_i + \lambda_j) G_{ij}$$

- Kořeny: $\phi_j \in g_0^*$, $\phi_j(H) = \lambda_j$:

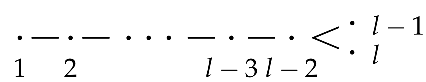
$$\Delta = \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i \neq j\} \cup \{-(\phi_i + \phi_j) | i < j\}$$

$H_0 = \operatorname{diag}(\lambda_1, \dots, \lambda_l)$, $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$:

$$\begin{aligned} \Delta^+ &= \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i < j\} \\ \Delta^p &= \{\underbrace{\phi_i - \phi_{i+}}_{=:\alpha_i} | i \in \widehat{l-i}\} \cup \{\underbrace{\phi_{l-1} + \phi_l}_{=:\alpha_l}\} \end{aligned}$$

$$\begin{array}{llll} \alpha_i + k\alpha_{i+1} & = (\phi_i - \phi_{i+1}) + k(\phi + 1 - \phi_{i+2}),\ i \in \widehat{l-1} & \Rightarrow & k = 0 \vee k = 1 \\ \alpha_{l-2} + k\alpha_l & = (\phi_{l-2} - \phi_{l-1}) + k(\phi_{l-1} + \phi_l) & \Rightarrow & k = 0 \vee k = 1 \\ \alpha_{l-1} + k\alpha_l & = (\phi_{l-1} - \phi_l) + k(\phi_{l-1} + \phi_l) & \Rightarrow & k = 0 \end{array}$$

$$(a_{ij}) = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & 2 & -1 & -1 \\ & & -1 & 2 & 0 \\ & & -1 & 0 & 2 \end{pmatrix},$$



- Váhy:

$$H = \begin{pmatrix} d_1\sigma_2 & & \\ & \ddots & \\ & & d_l\sigma_2 \end{pmatrix} = H(d_1, \dots, d_l), \quad \begin{aligned} &\phi_i(H) = d_i \\ &\alpha_i = \phi_i - \phi_{i+1},\ i \leqslant l-1 \\ &\alpha_l = \phi_{l-1} + \phi_l \\ &T_i = H(0, \dots, 0, \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 0),\ i \leqslant l - 1x \end{aligned}$$

T_l:

$$\left.\begin{array}{rll} \alpha_{l-2}(T_l) & = -1 & = d_{l-2} - d_{l-1} \\ \alpha_{l-1}(T_l) & = 0 & = d_{l-1} - d_l \\ \alpha_l(T_l) & = 2 & = \phi_{l-1}(T_l) + \phi_l(t_l) = d_{l-1} + d_l \end{array}\right\} \quad \Rightarrow \quad T_l = H(0, \dots, 0, 1, 1)$$

$\lambda_i(T_j) = \delta_{ij}$:

$$\begin{aligned} \lambda_1 &= \phi_1 \\ \lambda_i &= \phi_1 + \cdots + \phi_i,\ i \leqslant l-2 \\ \lambda_{l-1} &= \frac{1}{2}(\phi_1 + \cdots + \phi_{l-1} - \phi_l) \\ \lambda_l &= \frac{1}{2}(\phi_1 + \cdots + \phi_l) \end{aligned}$$

Definující reprezentace má váhy $\{\phi_1, \dots, \phi_l, -\phi_1, \dots, -\phi_l\}$.

Příklad 5. $B_l = \mathfrak{so}(2l+1, \mathbb{C})$

$$\mathfrak{g}_0 = \left\{ H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} \right\}, \quad \phi_i H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} = \lambda_i, \quad X := \left(\begin{array}{c|c} & v \\ \hline v^T & 0 \end{array}\right)$$

$$
[H,X] = \left(\begin{array}{c|c} & \lambda_i \sigma_1 v \\ \hline & 0 \end{array}\right) - \left(\begin{array}{c|c} & v \\ \hline -\lambda_i(\sigma_1 v)^T & 0 \end{array}\right) = \lambda_i \left(\begin{array}{c|c} & \sigma_1 v \\ \hline (\sigma_1 v)^T & 0 \end{array}\right)
$$

Za v můžeme volit vlastní vektory σ_1. Dále zvolíme $H_0 : \; \lambda_1 > \cdots > \lambda_l, \; \lambda_i = \phi(H_0)$.

$$
\begin{aligned}
\Delta &= \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i \neq j\} \cup \{-(\phi_i + \phi_j) | i < j\} \cup \{\phi_i\} \cup \{-\phi_i\} \\
\Delta^+ &= \{\phi_i + \phi_j | i < j\} \cup \{\phi_i - \phi_j | i < j\} \cup \{\phi_i\} \\
\Delta^p &= \{ \underbrace{\phi_i - \phi_{i+1}}_{=:\alpha_i} | i \in \widehat{l-1}\} \cup \{ \underbrace{\phi_l}_{=:\alpha_l} \}
\end{aligned}
$$

$$
\begin{array}{lll}
\alpha_{l-2} + k\alpha_l & = (\phi_{l-2} - \phi_{l-1}) + k\phi_l & \Rightarrow \quad k = 0 \\
\alpha_{l-1} + k\alpha_l & = (\phi_{l-1} - \phi_l) + k\phi_l & \Rightarrow \quad k = 0 \vee k = 1 \vee k = 2 \\
\alpha_l + k\alpha_{l-1} & = \phi_l + k(\phi_{l-1} - \phi_l) & \Rightarrow \quad k = 0 \vee k = 1
\end{array}
$$

$$
(a_{ij}) = \begin{pmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & 2 & -1 & & \\ & & -1 & 2 & -2 & \\ & & & -1 & 2 & \end{pmatrix}, \qquad \begin{array}{ccccccccc} \bullet & - & \bullet & - & \cdots & - & \bullet & - & \bullet \Leftarrow \bullet \\ 1 & & 2 & & & & l-2 & & l-1 \quad\; l \end{array}
$$

$$
H = \begin{pmatrix} d_1\sigma_2 & & & \\ & \ddots & & \\ & & d_l\sigma_2 & \\ & & & 0 \end{pmatrix} \qquad
\begin{array}{l}
\phi_i(H) = d_i \\
\alpha_i = \phi_i - \phi_{i+1}, \; i \leqslant l-1 \\
\alpha_l = \phi_l \\
T_i = H(0, \dots, 0, \underset{i}{1}, \underset{i+1}{-1}, 0, \dots, 0)
\end{array}
$$

T_l:

$$
\left.\begin{array}{rl} \alpha_{l-1}(T_l) & = -2 \\ \alpha_l(t_l) & = 2 \end{array}\right\} \quad \Rightarrow \quad T_l = H(0, \dots, 0, 2)
$$

$\lambda_i(T_j) = \delta_{ij}$:

$$
\begin{aligned}
\lambda_i &= \phi_1 + \cdots + \phi_i, \; i \leqslant l-1 \\
\lambda_l &= \frac{1}{2}(\phi_1 + \cdots + \phi_l)
\end{aligned}
$$