1 Reálné formy komplexních poloprostých algeber

Mějme poloprostou reálnou algebru $\mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus_{\mathbb{R}} i\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. Z $\mathfrak{g}_{\mathbb{C}}$ lze zpetně najít $\mathfrak{g}$:

$$\phi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}} : \phi(u+iv) = u - iv,\ \forall u,v \in \mathfrak{g} \quad \Rightarrow \quad \mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} | \phi(X) = X\}.$$

Poznámka 1. Vlastnosti ϕ:

1. $\phi \circ \phi = \mathrm{id} \quad \Rightarrow \quad$ je involutivní,
2. $\phi(\lambda X) = \overline{\lambda}\phi(X)$, $\phi(X+Y) = \phi(X) + \phi(Y) \quad \Rightarrow \quad$ je antilineární,
3. $\phi\big([u_1 + iv_1, u_2 + iv_2]\big) = \phi\big(\,([u_1,u_2] - [v_1,v_2]) + i\,([v_1,u_2] + [u_1,v_2])\,\big) = ([u_1,u_2] - [v_1,v_2]) - i\,([v_1,u_2] + [u_1,v_2]) = [u_1 - iv_1, u_2 - iv_2] = \big[\phi(u_1+iv_1), \phi(u_2+iv_2)\big] \quad \Rightarrow \quad$ automorfismus.

Tj. reálná forma $\mathfrak{g}$ komplexní algebry $\mathfrak{g}_{\mathbb{C}}$ nám určuje involutivní antilineární automorfismus ϕ.

Naopak, mějme $\mathfrak{g}_{\mathbb{C}}$ a její involutivní antilineární automorfismus ϕ. Pak $\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} | \phi(X) = X\}$ nám zadává reálnou podalgebru $\mathfrak{g}_\phi$ v $\mathfrak{g}_{\mathbb{C}} : (\mathfrak{g}_\phi)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$.

Důkaz. Máme $\phi : \mathfrak{g}_{\mathbb{C}} \to (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$, $\dim_{\mathbb{R}}(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} = 2\dim_{\mathbb{C}}(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$. Uvažujme $\phi|_{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}} : (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} \to (\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$, $\phi^2 = \mathbb{1} \quad \Rightarrow \quad \sigma\left(\phi|_{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}}\right) = \pm 1$, takže

$$(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}} = \underbrace{\ker\left(\phi|_{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}} - \mathbb{1}\right)}_{\mathfrak{g}} \dotplus \ker\left(\phi|_{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}} + \mathbb{1}\right)$$

$\Rightarrow \quad X \in \mathfrak{g} \quad \Rightarrow \quad \phi(X) = X \quad \Rightarrow \quad \phi(iX) = -iX \quad \Rightarrow \quad iX \in \ker\left(\phi|_{(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}} + \mathbb{1}\right) \quad \Rightarrow \quad$ obě jádra mají stejnou dimenzi a násobení i zobrazuje jedno na druhé $\quad \Rightarrow \quad \dim_{\mathbb{R}} \mathfrak{g} = \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ a platí: $\forall X, Y \in \mathfrak{g}$, $\phi\big([X,Y]\big) = [\phi(X), \phi(Y)] = [X,Y] \quad \Rightarrow \quad \forall X, Y \in \mathfrak{g}$, $[X,Y] \in \mathfrak{g}$. ☐

Příklad 1. $\mathfrak{sl}(l+1,\mathbb{C})$, $\phi : \mathfrak{sl}(l+1,\mathbb{C}) \to \mathfrak{sl}(l+1,\mathbb{C})$:

- $\phi(A) = \overline{A}\ \ldots\ \mathfrak{g} = \mathfrak{sl}(l+1,\mathbb{R})$
- $\phi(A) = -A^+\ \ldots\ \mathfrak{g} = \big\{X \in \mathfrak{sl}(l+1,\mathbb{C}) \big| -X^+ = X\big\} = \mathfrak{su}(l+1)$
- $\phi(A) = -JA^+J$, kde $J = \mathrm{diag}(\underbrace{1,\ldots 1}_{p}, \underbrace{-1,\ldots,-1}_{q})$:

$$\begin{aligned}\phi(\phi(A)) &= J\left(J\left(A^+\right)^+ J\right)J = A\\ \phi\big([A,B]\big) &= -J[A,B]^+J = J\left[A^+, B^+\right]J = \left[-JA^+J, -JB^+J\right] = [\phi(A), \phi(B)]\end{aligned}$$

$\ldots\ \mathfrak{g} = \big\{X \in \mathfrak{sl}(l+1,\mathbb{C}) \big| -JX^+J = X\big\} = \mathfrak{su}(p,q),\ p + q = l+1.$

Uvažujme komplexní poloprostou Lieovu algebru $\mathfrak{g}$ vyjádřenou ve Weyl-Chevalleyho bázi, tj.:

$$\begin{gathered}\mathfrak{g} = \mathrm{span}\{H_\alpha\}_{\alpha\in\Delta^p} \dotplus \dot{\sum_{\alpha\in\Delta}} \mathrm{span}\{E_\alpha\},\\ [H, E_\alpha] = \alpha(H)E_\alpha,\ H \in \mathrm{span}_{\mathbb{R}}\{H_\alpha\}_{\alpha\in\Delta^p} = \mathfrak{h} \quad \Rightarrow \quad \forall\alpha \in \Delta,\ \alpha(H) \in \mathbb{R}\\ [E_\alpha, E_{-\alpha}] = \underbrace{K(E_\alpha, E_{-\alpha})}_{\in\mathbb{R}} H_\alpha,\\ [E_\alpha, E_\beta] = N_{\alpha\beta}E_{\alpha+\beta},\ \mathbb{N}_{\alpha\beta} \in \mathbb{Z},\ N_{(-\alpha)(-\beta)} = -N_{\alpha\beta}.\end{gathered}$$

Označíme

$$\mathfrak{g}_{\text{split}} := \operatorname{span}_{\mathbb{R}}\{H_\alpha\}_{\alpha\in\Delta^p} \dotplus \dot{\sum}_{\alpha\in\Delta}\operatorname{span}_{\mathbb{R}}\{E_\alpha\}.$$

$\mathfrak{g}_{\text{split}}$ je reálná forma $\mathfrak{g}$, tj. $\phi(H_\alpha) = H_\alpha$, $\phi(E_\alpha) = E_\alpha$, $\forall H_\alpha, E_\alpha \in \mathfrak{g}_{\text{split}}$,. Zjistíme signaturu Killingovy formy K pro $\mathfrak{g}_{\text{split}}$ (Killingova forma je dobrá pro rozlišení reálnych algeber).

$$\begin{aligned}
&K(H, E_\alpha) = 0 \text{ protože } \mathfrak{h} \perp \operatorname{span}\{E_\alpha\}\\
&K|_{\mathfrak{h}} \text{ je pozitivně definitní}\\
&K(E_\alpha, E_\beta) = 0,\ \alpha+\beta \neq 0\\
&E_\alpha, E_{-\alpha} : \begin{pmatrix} K(E_\alpha, E_\alpha) & K(E_\alpha, E_{-\alpha}) \\ K(E_{-\alpha}, E_\alpha) & K(E_{-\alpha}, E_{-\alpha}) \end{pmatrix} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \text{kde } \lambda \neq 0 \ \ldots\ \operatorname{sgn} = (1,1,0)
\end{aligned}$$

$\Rightarrow \quad \operatorname{sgn} K|_{\mathfrak{g}_{\text{split}}} = \left(l + \frac{n-l}{2}, \frac{n-l}{2}, 0\right)$. Dále prozkoumáme

$$\mathfrak{g}_{\text{komp}} := \underbrace{\operatorname{span}_{\mathbb{R}}\{iH_\alpha\}_{\alpha\in\Delta^p}}_{i\mathfrak{h}} \dotplus \dot{\sum_{\alpha\in\Delta^+}} \operatorname{span}\left\{\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}, \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\right\}.$$

$\forall H_\alpha, E_\alpha \in \mathfrak{g}_{\text{komp}}$, $\phi(H_\alpha) = -H_\alpha$, $\phi(E_\alpha) = -E_{-\alpha}$. Zřejmě platí taky $\phi^2 = \mathbb{1}$. Dále máme:

$$\begin{aligned}
&\phi\big([H_\alpha, H_\beta]\big) = [\phi(H_\alpha), \phi(H_\beta)] \text{ protože } [H_\alpha, H_\beta] = 0\\
&\phi\big([H_\alpha, E_\beta]\big) = \underbrace{\beta(H_\alpha)}_{\in\mathbb{R}}\phi(E_\beta) = -\beta(H_\alpha)E_{-\beta} = [H_\alpha, E_{-\beta}] = [\phi(H_\alpha), \phi(E_\beta)]\\
&\phi\big([E_\alpha, E_\beta]\big) = N_{\alpha\beta}\phi(E_{\alpha+\beta}) = -N_{\alpha\beta}E_{-\alpha-\beta} = N_{(-\alpha)(-\beta)}E_{-\alpha-\beta} = [-E_{-\alpha}, -E_{-\beta}] =\\
&\qquad = [\phi(E_\alpha), \phi(E_\beta)]\\
&\phi\big([E_\alpha, E_{-\alpha}]\big) = K(E_\alpha, E_{-\alpha})\phi(H_\alpha) = -K(E_\alpha, E_{-\alpha})H_\alpha = -[E_\alpha, E_{-\alpha}] = [-E_{-\alpha}, -E_\alpha]\\
&\qquad = [\phi(E_\alpha), \phi(E_{-\alpha})]\\
&\phi\left(\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}\right) = \frac{-E_{-\alpha} + E_\alpha}{\sqrt{2}}\\
&\phi\left(\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\right) = \frac{-i(-E_{-\alpha} - E_\alpha)}{\sqrt{2}} = \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\\
&\phi(iH_\alpha) = -i(-H_\alpha) = iH_\alpha\\
&K|_{i\mathfrak{h}} \text{ je negativně definitní}\\
&\left.\begin{aligned}
&K\left(\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}, \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}\right) = -K(E_\alpha, E_{-\alpha})\\
&K\left(\frac{E_\alpha - E_{-\alpha}}{\sqrt{2}}, \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\right) = 0\\
&K\left(\frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}, \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}\right) = -K(E_\alpha, E_{-\alpha})
\end{aligned}\right\}\operatorname{sgn} = (0,2,0) \text{ pro volbu } K(E_\alpha, E_{-\alpha}) > 0
\end{aligned}$$

$\Rightarrow \quad \operatorname{sgn} K|_{\mathfrak{g}_{\text{kompl}}} = (0, n, 0)$.

Věta 1. (Weyl) Buď $\mathfrak{g}$ reálná poloprostá Lieova algebra, G jí odpovídající souvislá a jednoduše souvislá Lieova grupa. Pak G je kompaktní $\Leftrightarrow$ Killingova forma $\mathfrak{g}$ je negativně definitní. Bez důkazu.